Mathematical analysis of fully coupled approach to creep damage

A.V. Shutov, A.-M. Saendig
Institute for Applied Analysis and Numerical Simulation
70569 Stuttgart, Germany

shutov@ngs.ru,saendig@ians.uni-stuttgart.de

Abstract

We prove the existence and uniqueness of solution to a classical creep damage problem. We formulate a sufficient condition for the problem to have a unique smooth solution, locally in time. This condition is stated in terms of smoothness of given data, such as solid geometry, boundary conditions, applied loads, and initial conditions. Counterexamples with an arbitrary small lifetime of a structure are also given, showing the mechanical interpretation of imposed smoothness conditions. The proposed theory gives a rigorous framework for a strain localization analysis. The influence of the damage gradient on the strain localization process is characterized within this framework and a measure of the damage localization is proposed.

Key words: continuum damage mechanics, Kachanov-Rabotnov approach, creep, damage localization, well-posed problem, Sobolev space

AMS Subject Classification: 74A45, 74E05, 74G30, 74G70, 74R99

1 Introduction

Structures made of metals and alloys are often used in different brunches of industry at elevated temperatures (higher than 0.3 times the melting temperature). Typical examples are pressurized pipes and vessels in power and chemical plants, gas turbines and so on. Even subjected to moderate loads these structures experience irreversible creep deformations which influence the stress response in long time scales. The lifetime of such structures is limited by damage processes induced by the nucleation and growth of microscopic cracks and cavities. The finite element method (FEM) is commonly used for numerical analysis of nonlinear creep-damage response in the framework of continuum damage mechanics (see [10]) to estimate the remaining lifetime. The classical creep models are described and analyzed in the following monographs [13], [16], [14], [27].

The creep behavior is divided into three stages. The initial stage is characterized by hardening behavior with decreasing creep strain rate. The second stage is the

stationary creep with a constant creep strain rate. The last stage is the tertiary creep characterized by increasing creep strain rate and a dominant softening of the material followed by a complete rupture. The most popular constitutive law for the second stage was proposed by Norton [22] and postulates the stationary creep rate as a power law function of the stress tensor. This constitutive law is modified by use of time or hardening parameters (see, for example, [27], [17]) to take the primary creep into account. A new internal continuity parameter ψ was introduced in the original work of Kachanov [15] to simulate the material damage within the tertiary creep. This continuity parameter is often replaced by a dual variable, namely, Rabotnov's damage parameter $\omega = 1 - \psi$ (see [13]). Within Kachanov-Rabotnov's approach the damage rate is postulated as a function of the stress, the temperature and the current damage state. This is regarded as a foundation of continuum damage mechanics (CDM).

One of unsolved problems of computational CDM is the spurious mesh-dependence of FEM simulations (compare [18], [20], [3], [24], [25]) which leads to physically unrealistic results. Therefore numerous regularization techniques were proposed to prevent this mesh-dependence (see, for example, [23], [9], [20]).

Proposed material models and regularization techniques are generally tested by series of numerical experiments. At the same time the mathematical treatment of nonlinear material models is very poor. Some mathematical results are given in [2], [4], [5], [21]. In [2] the local existence and uniqueness is proved for a coupled creep-damage model assuming the elastic properties are not influenced by damage evolution (partly coupled approach).

For most of the used damage models it is not clear whether the corresponding boundary value problems are well-posed. We say that a given problem is well-posed (see, for example, [6]) if the problem in fact has a solution; this solution is unique; and the solution depends continuously on the data given in the problem. In case of a creep-damage problem such given data are solid geometry, boundary conditions, applied loads, initial conditions, and material constants. A mathematically consistent problem statement is necessary for justification of analytical (see the paper [26]) and numerical techniques. Particulary it specifies how the difference between exact and approximate solution can be measured and what kind of perturbations of given data are allowed.

The proper mathematical analysis of nonlinear damage models is complicated by instabilities due to loss of ellipticity of the corresponding differential operator (compare [12], [11], [2] for example). On the other hand, bifurcation does not happen before the appearance of completely damaged zone with $\omega = \omega^*$, where ω^* is a critical damage value (see, for example, [18]). The period of time required by the structure to reach this state is called crack growth initiation time t^*. Therefore we prove existence and uniqueness of solution in sufficiently small time interval before failure initiation. On this time interval the deformation is stable and the problem can be posed correctly. Thus, the analysis of crack propagation lies beyond the scope of this article.

The article is organized so that technical details of the proof do not obscure the main points. First, we introduce an initial boundary value problem for fully coupled creep-damage model. In the following section we give basic definitions of

function spaces, that are necessary for the formulation of the main result. In section 4 the main existence and uniqueness theorem is formulated. One counterexample is provided, which illustrates the effect of damage localization. Finally we prove the main result and summarize our main conclusions.

2 Constitutive equations

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain which represents the solid. In this work we confine ourselves to the plane-stress two-dimensional case. But the theory proposed here can be easily generalized to three dimensions. Let us assume a stationary temperature field. Therefore the constitutive equations do not depend on temperature.

2.1 Fully coupled damage model

Suppose that the damage evolution is controlled by the von Mises equivalent stress. Accordingly to the classical Kachanov-Rabotnov concept the constitutive equations for secondary and tertiary creep are summarized as follows

$$\boldsymbol{\sigma} = \mathbf{C}^{\omega}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{cr}) \quad \text{in } \Omega \times [0, T], \tag{1}$$

$$\dot{\boldsymbol{\varepsilon}}^{cr} = \frac{3}{2} A \ \mathbf{s} \ (\sigma_{vM})^{n-1} (1 - \omega)^{-n} \quad \text{in } \Omega \times (0, T), \tag{2}$$

$$\dot{\omega} = B \ (\sigma_{vM})^{m} (1 - \omega)^{-q} \quad \text{in } \Omega \times (0, T), \tag{3}$$

$$\mathbf{s} = \boldsymbol{\sigma} - \frac{1}{3} \mathrm{tr}(\boldsymbol{\sigma}) \mathbf{I}, \quad \sigma_{vM} = \sqrt{\frac{3}{2} \mathbf{s} : \mathbf{s}}, \tag{4}$$

where $\boldsymbol{\sigma}$ is the stress tensor, $\mathbf{C}^{\omega}$ is the fourth-rank tensor depending on Rabotnov's damage parameter ω, $\dot{(\ )}$ is the time derivative, $\boldsymbol{\varepsilon}^{cr}$ is the creep strain, $\mathbf{s}$ is the stress deviator, σ_{vM} is the von Mises equivalent stress, $\mathbf{I}$ is the second rank unit tensor, and A, B, n, m, q are material constants. The influence of the damage on elastic properties is given by the equation (see [19], [18])

$$\mathbf{C}^{\omega} = \mathbf{C}(1 - \omega). \tag{5}$$

Here $\mathbf{C}$ denotes the tensor of linear elasticity of undamaged solid. $\mathbf{C}$ is linear, symmetric, positive definite mapping.

Furthermore, we consider equilibrium equations

$$\nabla \cdot \boldsymbol{\sigma} = -\mathbf{q} \quad \text{in } \Omega \times [0, T], \tag{6}$$

and strain-displacement relations

$$\boldsymbol{\varepsilon} = \frac{1}{2} \left(\nabla \mathbf{u} + \nabla \mathbf{u}^{T} \right), \tag{7}$$

where $\boldsymbol{\varepsilon}$ is the linearized strain tensor and $\mathbf{u}$ is the displacement vector. The quantities $(\mathbf{u}, \boldsymbol{\varepsilon}^{cr}, \omega, \boldsymbol{\sigma})$ depend on the space variable $x \in \Omega$ and the time variable $t \in [0, T]$ for some $T > 0$. The system is completed by boundary and initial conditions

$$\mathbf{u} = \mathbf{u}^* \quad \text{on } \partial\Omega \times [0, T], \tag{8}$$

$$\boldsymbol{\varepsilon}^{cr}|_{t=0} = \boldsymbol{\varepsilon}_0^{cr} \quad \text{in } \Omega, \tag{9}$$

$$\omega|_{t=0} = \omega_0 \quad \text{in } \Omega. \tag{10}$$

Remark. The constitutive equation for damage evolution (3) was generalized by Hayhurst (see [11]) so that $\dot{\omega}$ depends on the combination $\alpha\sigma_1 + (1 - \alpha)\sigma_{vM}$ of the maximal principal stress and von Mises equivalent stress. But the proof of the main result (*Theorem 4.1*) is essentially based on the smoothness of constitutive equations, therefore we do not analyze this popular model here.

2.2 Remark on the partly coupled damage model

Equation (5) of the fully coupled model is based on Kachanov's concept of reduction of the effective load carrying area. This equation is a fundamental form of elastic-damage coupling.

Within the partly coupled approach the influence of damage on the elastic properties is neglected and equation (5) is replaced by

$$\mathbf{C}^\omega = \begin{cases} \mathbf{C} & \text{if } \omega < \omega^* \\ 0 & \text{if } \omega = \omega^* \end{cases}, \tag{11}$$

where ω^* stands for critical damage state.

In most of engineering applications this approach is used as a simplified variant of the fully coupled relations in order to decrease the computational effort. Unlike the fully coupled model, the partly coupled approach does not require a modification and decomposition of the stiffness matrix on each time or iteration step. As it was observed in [18] and [3], the partly coupled approach gives a good estimation of failure time for some specimens and initial conditions. Nevertheless, as it will be shown later, this simplification should be used carefully. This model does not take into account the stress concentrations caused by damage inhomogeneity. For instance, if $A = 0$ in (2) and $B \neq 0$ in (3), then the partly coupled system (11) describes linear elasticity of homogeneous solid.

Existence and uniqueness for partly coupled model were proved in [2] in case of thin-walled structures. Thus, the plane stress was covered as a special case of shell geometry. We generalize the existence proof, given in [2], to take the fully coupled damage model into account.

3 Basic notations

The creep-damage problem (1) — (10) can be formulated in a well-posed manner with the help of suitable function spaces. Field variables which describe the structure

are considered to be elements of these infinite dimensional spaces. The corresponding function norms should take into account the physical essence of the problem and the properties of the system of equations.

3.1 Definition of function spaces

Let B be a Banach space endowed with a norm $\|\cdot\|_B$ and T be a positive real number. We introduce a space of continuous B-valued functions defined on the interval $[0,T]$.

Definition 1

$$C^0([0,T],B) := \{\varphi : [0,T] \to B,\ \varphi \text{ is continuous}\}. \tag{12}$$

This space is a Banach space equipped with the norm

$$\|u\|_{B,\infty} = \sup\{\|u(t)\|_B : t \in [0,T]\}. \tag{13}$$

Let $k \in \mathbb{N}_0$ and $p \geqslant 1$. We define the usual Sobolev space $W^{k,p}(\Omega)$ (see, for example, [1], [6], [7]).

Definition 2

$$W^{k,p}(\Omega) := \{u \in L_p : D^\alpha u \in L_p \text{ for all } \alpha = (\alpha_1,\alpha_2) \in \mathbb{N}_0^2, \alpha_1 + \alpha_2 \le k\} \tag{14}$$

endowed with the norm

$$\|u\|_{k,p} = \Big(\sum_{\alpha\in\mathbb{N}_0^2,\ |\alpha|\le k} \|D^\alpha u\|_p^p\Big)^{\frac{1}{p}}. \tag{15}$$

Here $D^\alpha u$ are generalized derivatives of the order $|\alpha| = \alpha_1 + \alpha_2$.

Beside the Sobolev space $W^{k,p}$ we will need a proper subspace $W_0^{k,p}(\Omega) \subset W^{k,p}$ which is defined as follows.

Definition 3

Let $C_0^\infty(\Omega) := \{\varphi \in C^\infty(\Omega), \mathrm{supp}\ \varphi \subset \Omega\}$ be the set of smooth functions that vanish near the boundary $\partial\Omega$. Then

$$W_0^{k,p}(\Omega) := \overline{C_0^\infty(\Omega)}^{W^{k,p}(\Omega)} \tag{16}$$

is the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{k,p}$. Note that functions from $W_0^{k,p}(\Omega)$ vanish on the boundary in the trace sense (see *Definition 4*).

Theorem 3.1 (Imbedding theorem, see Theorem 7.26 in [7])

Let $p > 2$ and Ω be a Lipschitz domain in $\mathbb{R}^2$. Then $W^{k,p}(\Omega)$ is continuously imbedded in $C^{0,k-2/p}(\overline{\Omega})$.

Corollary 3.1 (Sobolev inequality)

Let $p > 2$ and Ω be a Lipschitz domain in $\mathbb{R}^2$. Then there is a constant $C_I < \infty$ with

$$\|u\|_{C^0(\overline{\Omega})} \le C_I \|u\|_{W^{1,p}(\Omega)}. \tag{17}$$

Furthermore we need the traces of functions from $W^{2,p}(\Omega)$ on the boundary $\partial\Omega$.

Definition 4

Let Ω be a bounded domain with $C^{1,1}$-boundary. That means the boundary $\partial\Omega$ is locally given by a function with a Lipschitz continuous derivative. Suppose $p > 1$. Then the trace space of functions from $W^{2,p}(\Omega)$ is defined as (see [8], pp. 37-38)

$$W^{2-\frac{1}{p},p}(\partial\Omega) = \{u \in W^{1,p}(\partial\Omega) : \int\limits_{\partial\Omega}\int\limits_{\partial\Omega} \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^p} ds_x ds_y < \infty$$
$$\text{for all } \alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2, \alpha_1 + \alpha_2 \leq 1\}. \quad (18)$$

If $p > 1$, then the trace operator

$$\mathrm{Tr} : W^{2,p}(\Omega) \to W^{2-\frac{1}{p},p}(\partial\Omega), \quad (19)$$

$$\mathrm{Tr} : u \mapsto u|_{\partial\Omega} \quad (20)$$

is well defined in the classical sense.

If the time t is fixed, then we consider the field variables to be the functions, which are defined in Ω and belong to the proper function spaces. We will use the following abbreviations of function spaces and subsets:

- $X_p := (L_p(\Omega))^2$ for the volumetric loads,
- $Y_p := W^{1,p}(\Omega)$ for the components of creep strain tensor,
- $Y_p^4 := (Y_p)^4$ for the creep strain tensors,
- $V_p := (W^{2,p}(\Omega))^2$ for the displacement fields,
- $V_p^0 := (W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega))^2$ for the displacement fields with a vanishing boundary values (which correspond to the solid clamped at the boundary),
- $Y_p^{\beta_1,\beta_2} := \{\omega \in Y_p : 0 \leq \omega(x) \leq 1 - \beta_1, \|\omega\|_{Y_p} \leq \beta_2\}$ for the damage fields, where $\frac{1}{2} > \beta_1 > 0, \beta_2 > 0$ are fixed constants. Accordingly to *Corollary 3.1*, $\omega(x)$ is well defined and $Y_p^{\beta_1,\beta_2}$ is a closed subset of Y_p.

Remark. The condition $0 \leq \omega(x) \leq 1 - \beta_1$ is natural to guarantee that the elasticity tensor (5) is positive definite. The second condition $\|\omega\|_{Y_p} \leq \beta_2$ imposes additional constraints both on the damage field and on the damage gradient.

3.2 Reduction to zero prescribed displacements

In this subsection we reduce the boundary value problem (1) — (10) to the case of zero prescribed displacements along the boundary $\partial\Omega$.

Theorem 3.2

Suppose that displacements, which are given on the boundary, satisfy the following smoothness condition: $\mathbf{u}^* \in (W^{2-\frac{1}{p},p}(\partial\Omega))^2$. *Then there is a function* $\hat{\mathbf{u}} \in V_p$ *with* $\hat{\mathbf{u}}|_{\partial\Omega} = \mathbf{u}^*$ *in the trace sense* (see [8]).

In what follows, we designate $\hat{\mathbf{u}}$ by the same symbol as $\mathbf{u}^*$.

Now we reformulate our problem in a standard way. We search a vector $(\mathbf{u}, \boldsymbol{\varepsilon}^{cr}, \omega) \in V_p^0 \times Y_p^4 \times Y_p$, such that $(\mathbf{u} + \mathbf{u}^*, \boldsymbol{\varepsilon}^{cr}, \omega) \in V_p \times Y_p^4 \times Y_p$ is a solution of (1) — (10).

3.3 Compact form of evolution equations

In this subsection we rewrite the evolution equations (2), (3) in a compact form

$$\dot{\boldsymbol{\varepsilon}}^{cr}(x,t) = \mathcal{R}(\rho(\mathbf{u}, \boldsymbol{\varepsilon}^{cr}, \omega)(x,t)) \tag{21}$$

$$\dot{\omega}(x,t) = \mathcal{S}(\rho(\mathbf{u}, \boldsymbol{\varepsilon}^{cr}, \omega)(x,t)). \tag{22}$$

To this end we introduce for every $(\mathbf{u}, \boldsymbol{\varepsilon}^{cr}, \omega) \in V_p \times Y_p^4 \times Y_p^{\beta_1,\beta_2}$,

$$\rho(\mathbf{u}, \boldsymbol{\varepsilon}^{cr}, \omega) := \left(\varepsilon_{11}(\mathbf{u}), \varepsilon_{22}(\mathbf{u}), \varepsilon_{12}(\mathbf{u}), \varepsilon_{11}^{cr}, \varepsilon_{22}^{cr}, \varepsilon_{12}^{cr}, \omega\right) \in Y_p^7, \tag{23}$$

$$\boldsymbol{\varepsilon}(\mathbf{u}) := \frac{1}{2}\left(\boldsymbol{\nabla}\mathbf{u} + \boldsymbol{\nabla}\mathbf{u}^T\right) \in Y_p^4. \tag{24}$$

For every $\rho \in \mathbb{R}^7$ we define

$$\mathcal{R}(\rho) := \frac{3}{2}A\ \mathbf{s}(\rho)\ (\sigma_{vM}(\rho))^{n-1}(1-\rho_7)^{-n}, \tag{25}$$

$$\mathcal{S}(\rho) := B\ (\sigma_{vM}(\rho))^{m}(1-\rho_7)^{-q}, \tag{26}$$

$$\sigma_{vM}(\rho) := P(\sigma_{11}(\rho), \sigma_{22}(\rho), \sigma_{12}(\rho)), \tag{27}$$

$$\mathbf{s}(\rho) := \boldsymbol{\sigma}(\rho) - \frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma}(\rho))\mathbf{I}, \tag{28}$$

$$\boldsymbol{\sigma}(\rho) := \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}(\rho), \tag{29}$$

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix}(\rho) := (1-\rho_7)\frac{E}{1-\nu^2}\begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{pmatrix}\begin{pmatrix} \rho_1 - \rho_4 \\ \rho_2 - \rho_5 \\ 2(\rho_3 - \rho_6) \end{pmatrix}, \tag{30}$$

$$P(z_1, z_2, z_3) := \sqrt{z_1^2 + z_2^2 - z_1 z_1 + 3z_3^2} \quad \text{for every} \quad z \in \mathbb{R}^3. \tag{31}$$

The constitutive relations (30), (31) are obtained from the general 3D relations under plane stress assumption ($\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$).

4 Main result

The existence and uniqueness theorem states that a unique smooth solution to the initial boundary value problem (1) — (10) exists in a certain time interval.

4.1 Formulation of the main theorem

Theorem 4.1

Let Ω be a bounded domain with $C^{1,1}$-boundary, $p > 2$, $T > 0$, $\mathbf{q} \in C^0([0, T], X_p)$, $\mathbf{u}^* \in V_p$, $\boldsymbol{\varepsilon}_0^{cr} \in Y_p^4$, and $\omega_0 \in Y_p^{\beta_1,\beta_2}$. Then there exists $T_1 \in (0, T]$ such that for any $T' \in (0, T_1]$ there is a uniquely determined mapping $(\mathbf{u}, \boldsymbol{\varepsilon}^{cr}, \omega) \in C^0([0, T'], V_p^0 \times Y_p^4 \times Y_p)$ such that

$$\boldsymbol{\nabla} \cdot \big((1 - \omega)\mathbf{C}(\varepsilon(\mathbf{u} + \mathbf{u}^*) - \boldsymbol{\varepsilon}^{cr})\big) = -\mathbf{q}(t) \quad \text{for all} \quad t \in [0, T'], \tag{32}$$

$$(\boldsymbol{\varepsilon}^{cr}, \omega)(t) = (\boldsymbol{\varepsilon}_0^{cr}, \omega_0) + \int_0^t \big(\mathcal{R}(\rho(\mathbf{u} + \mathbf{u}^*, \boldsymbol{\varepsilon}^{cr}, \omega), \mathcal{S}(\rho(\mathbf{u} + \mathbf{u}^*, \boldsymbol{\varepsilon}^{cr}, \omega)\big)(s)ds \tag{33}$$

for every $t \in [0, T']$. Here the evolution operators $\mathcal{R}, \mathcal{S}$ are defined by (21) — (31).

Moreover,

$$\omega(x, t) < 1 \quad \text{for all} \quad (x, t) \in \Omega \times [0, T'], \tag{34}$$

$$(\boldsymbol{\varepsilon}^{cr}, \omega) \in C^1([0, T'], Y_p^4 \times Y_p), \tag{35}$$

$$\dot{\boldsymbol{\varepsilon}}^{cr} = \mathcal{R}(\rho(\mathbf{u} + \mathbf{u}^*, \boldsymbol{\varepsilon}^{cr}, \omega), \quad \dot{\omega} = \mathcal{S}(\rho(\mathbf{u} + \mathbf{u}^*, \boldsymbol{\varepsilon}^{cr}, \omega), \tag{36}$$

$$\boldsymbol{\varepsilon}^{cr}(0) = \boldsymbol{\varepsilon}_0^{cr}, \quad \omega(0) = \omega_0, \tag{37}$$

$$\|\boldsymbol{\varepsilon}^{cr}(t) - \boldsymbol{\varepsilon}_0^{cr}\|_{Y_p^4} + \|\omega(t) - \omega_0\|_{Y_p} \leq \min\big(\frac{\beta_1}{2(1 + C_I)}, \frac{\beta_2}{2}\big) \; \forall \; t \in [0, T']. \tag{38}$$

Theorem 4.1 is proved in the next section.

Corollary 4.1

If the solid geometry, applied loads, prescribed displacements, and initial data are smooth, then the fully coupled creep-damage model predicts a nonzero lifetime t^* of the structure with a lower estimate T_1 from Theorem 4.1 ($t^* \geq T_1$).

Remark. Theorem 4.1 assures that $t^* \geq T_1 > 0$ only for smooth domains without notches. Numerous examples of FEM simulation of notched specimens show that the predicted crack initiation time t^* tends to zero as the mesh-size decreases ([18]).

Remark. The lifetime estimate T_1 depends on the constants β_1, β_2. Moreover, as it will be clear from the proof of *Theorem 4.1*, $T_1 = T_1(\beta_1/\beta_2)$. It is natural that $T_1 \to 0$ as $\beta_1 \to 0$ since the lifetime of the structure made of almost broken material $(\min\limits_{x\in\overline{\Omega}}(1 - \omega_0) \to 0)$ is negligibly small. Furthermore, T_1 tends also to zero as β_2 tends to infinity even if β_1 is finite. The physical interpretation of this result could be the following. The rupture time can be negligibly small in the case of big gradients of damage $(\|\omega_0\|_{Y_p} \to \infty)$ even if the initial damage itself was not substantial $(\min\limits_{x\in\overline{\Omega}}(1 - \omega_0) \sim 1)$.

Example is provided in the subsections 4.2 showing that the dependence of t^* on β_2 can be interpreted as lifetime reduction due to damage localization.

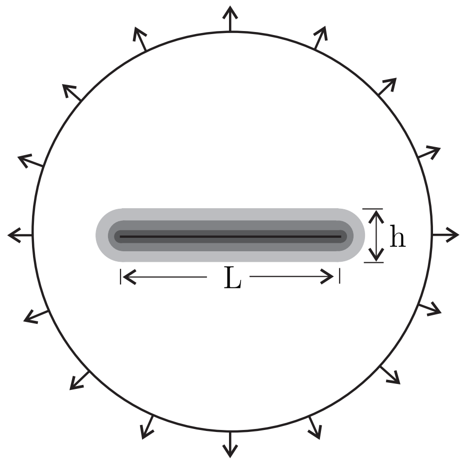

Figure 1: System configuration, boundary conditions, and initial damage

4.2 Counterexample: lifetime reduction due to local imperfections

Consider a solid loaded by prescribed displacements on it's boundary as shown on figure 1. Assume that the boundary and prescribed displacements are smooth. We set the creep strain rate to zero ($B = 0$ in (3)). Consider a curve l of length L within the solid. Suppose that initial damage is concentrated near the curve l (see fig. 1) and the initial creep is zero

$$\omega_0(x) := \max(0, \frac{h - \mathrm{dist}(x, l)}{2h}), \quad \varepsilon_0^{cr} := 0. \tag{39}$$

It is obvious that

$$\min(1 - \omega_0) \equiv 1/2, \quad \|\omega_0\|_{Y_p} \to \infty \quad \text{as} \quad h \to 0. \tag{40}$$

We assert that

$$t^* \to 0 \quad \text{as} \quad h \to 0. \tag{41}$$

To prove this assertion we can use the same argumentation as used in [18]. The main reason the lifetime is decreasing is because the stress concentration factor near the curve tip tends to infinity as $h \to 0$.

5 Proof of *Theorem 4.1*

To prove *Theorem 4.1* we need several lemmas.

5.1 Equilibrium equations with respect to u with a given ε^{cr} and ω

Define the family $\{\mathcal{L}_\omega\}_{\omega \in Y_p^{\frac{\beta_1}{2}, 2\beta_2}}$ of operators of linear elasticity

$$\mathcal{L}_\omega : V_p \to X_p \tag{42}$$

by the rule

$$\mathcal{L}_\omega(\mathbf{u}) := \boldsymbol{\nabla}\cdot\big((1-\omega)\mathbf{C}\varepsilon(\mathbf{u})\big) \tag{43}$$

for $\omega \in Y_p^{\frac{\beta_1}{2},2\beta_2}$, $\mathbf{u} \in V_p$.

Lemma 5.1

The operator $\mathcal{L}_\omega$ is bounded for all $\omega \in Y_p^{\frac{\beta_1}{2},2\beta_2}$. Moreover, the problem

$$\mathcal{L}_\omega(\mathbf{u}) = -\mathbf{q} \tag{44}$$

has a unique solution $\mathbf{u} \in V_p^0$ for all $\omega \in Y_p^{\frac{\beta_1}{2},2\beta_2}$, $\mathbf{q} \in X_p$ and

$$\|\mathbf{u}\|_{V_p} \le C_{5.1}\|\mathbf{q}\|_{X_p}. \tag{45}$$

Here $C_{5.1} < \infty$ does not depend on $\mathbf{q}$.

Proof. The boundness of $\mathcal{L}_\omega$ follows from the following computations

$$\begin{aligned}\|\mathcal{L}_\omega(\mathbf{u})\|_{X_p} &= \|\boldsymbol{\nabla}\cdot\big((1-\omega)\mathbf{C}\varepsilon(\mathbf{u})\big)\|_{X_p}\\ &\le \|\boldsymbol{\nabla}\omega\|_{X_p}\cdot\|\mathbf{C}\varepsilon(\mathbf{u})\|_{C^0} + \|1-\omega\|_{C^0}\cdot\|\boldsymbol{\nabla}\cdot\big(\mathbf{C}\varepsilon(\mathbf{u})\big)\|_{X_p}\\ &\le C(\beta_1,\beta_2)\|\mathbf{u}\|_{V_p}.\end{aligned} \tag{46}$$

Up to the rest of the article the expression $Q_1 \le C\cdot Q_2$ should be understood as follows. The quantities Q_1 and Q_2 are related to each other in such a way that there is a suitable constant $C < \infty$, which depends only on $(\Omega, E, \nu, p, n, m, q, \beta_1, \beta_2)$ and $Q_1 \le C\cdot Q_2$.

For the proof of solvability of (44) and for estimate (45) see [7] (p. 241). Particulary, we have the following inequality

$$\|\mathbf{u}\|_{V_p} \le C(\Omega, E, \nu, p)\frac{\beta_2}{\beta_1}\|\mathbf{q}\|_{X_p}. \tag{47}$$

The lemma is proved ■

Remark. The influence of the damage localization on the strain localization is taken into account by (47). Indeed,

$$C_{5.1} \to \infty, \quad \text{as } \frac{\beta_2}{\beta_1} \to \infty. \tag{48}$$

We denote by $\mathcal{L}_\omega|_{V_p^0}$ the restriction of $\mathcal{L}_\omega$ to V_p^0. Let $\mathcal{L}_\omega^{-1}$ be the inverse to $\mathcal{L}_\omega|_{V_p^0}$. Since (45) holds, we see that

$$\|\mathcal{L}_\omega^{-1}\| \le C_{5.1}. \tag{49}$$

Lemma 5.2

There is a constant $C_{5.2}$ such that

$$\|\mathcal{L}_{\omega_1} - \mathcal{L}_{\omega_2}\| \le C_{5.2}\|\omega_1-\omega_2\|_{Y_p}, \tag{50}$$

$$\|\mathcal{L}_{\omega_1}^{-1} - \mathcal{L}_{\omega_2}^{-1}\| \le C_{5.2}\|\omega_1-\omega_2\|_{Y_p}, \tag{51}$$

for all $\omega_1, \omega_2 \in Y_p^{\frac{\beta_1}{2}, 2\beta_2}$.

Proof. Obviously,

$$\|\mathcal{L}_{\omega_1} - \mathcal{L}_{\omega_2}\| = \sup_{\|\mathbf{u}\|_{V_p}=1} \|\nabla \cdot \big((\omega_2 - \omega_1)\mathbf{C}\varepsilon(\mathbf{u})\big)\|_{X_p}$$
$$\leq \sup_{\|\mathbf{u}\|_{V_p}=1} \Big(\|\nabla(\omega_1 - \omega_2)\|_{X_p} \cdot \|\mathbf{C}\varepsilon(\mathbf{u})\|_{C^0}$$
$$+ \|\omega_1 - \omega_2\|_{C^0} \cdot \|\nabla \cdot \big(\mathbf{C}\varepsilon(\mathbf{u})\big)\|_{X_p}\Big) \leq C\ \|\omega_1 - \omega_2\|_{Y_p}. \quad (52)$$

Furthermore,

$$\|\mathcal{L}_{\omega_1}^{-1} - \mathcal{L}_{\omega_2}^{-1}\| \leq \|\mathcal{L}_{\omega_1}^{-1}\| \cdot \|\mathcal{L}_{\omega_2}^{-1}\| \cdot \|\mathcal{L}_{\omega_1} - \mathcal{L}_{\omega_2}\| \leq C\ \|\omega_1 - \omega_2\|_{Y_p}. \quad (53)$$

The lemma is proved ■

Lemma 5.3

Let $\mathbf{u}^* \in V_p$, $T > 0$, $\mathbf{q} \in C^0([0,T], X_p)$, $\varepsilon^{cr} \in C^0([0,T], Y_p^4)$, $\omega \in C^0([0,T], Y_p^{\frac{\beta_1}{2}, 2\beta_2})$. Then there exists a uniquely determined mapping $\mathbf{U} = \mathbf{U}\big(\mathbf{u}^*, \varepsilon^{cr}, \omega, \mathbf{q}\big) \in C^0([0,T], V_p^0)$ such that

$$\mathcal{L}_{\omega(t)}(\mathbf{U}(t)) = -\mathcal{L}_{\omega(t)}(\mathbf{u}^*) - \mathbf{q}(t) + \nabla \cdot \big((1 - \omega(t))\mathbf{C}\varepsilon^{cr}(t)\big) \quad (54)$$

for $t \in [0,T]$. Moreover,

$$\|\mathbf{U}\big(\mathbf{u}^*, \varepsilon^{cr}, \omega, \mathbf{q}\big)\|_{V_p,\infty} \leq C_{5.3}\big(\|\mathbf{u}^*\|_{V_p} + \|\varepsilon^{cr}\|_{Y_p^4,\infty} + \|\mathbf{q}\|_{X_p,\infty}\big), \quad (55)$$

$$\|\mathbf{U}\big(\mathbf{u}^*, \varepsilon^{cr1}, \omega^1, \mathbf{q}^1\big) - \mathbf{U}\big(\mathbf{u}^*, \varepsilon^{cr2}, \omega^2, \mathbf{q}^2\big)\|_{V_p,\infty}$$
$$\leq C_{5.3}\Big(\|\omega^1 - \omega^2\|_{Y_p,\infty}\big(\|\mathbf{u}^*\|_{V_p} + \|\varepsilon^{cr1}\|_{Y_p^4,\infty} + \|\mathbf{q}^1\|_{X_p,\infty}\big)$$
$$+ \|\varepsilon^{cr1} - \varepsilon^{cr2}\|_{Y_p^4,\infty} + \|\mathbf{q}^1 - \mathbf{q}^2\|_{X_p,\infty}\Big). \quad (56)$$

Proof. We claim that the mapping $\mathbf{U} \in C^0([0,T], V_p^0)$ is uniquely defined by (54). Indeed, at each instant of time we have by *Lemma 5.1*

$$\mathbf{U}\big(\mathbf{u}^*, \varepsilon^{cr}, \omega, \mathbf{q}\big)(t) = \mathcal{L}_{\omega(t)}^{-1}\Big(-\mathcal{L}_{\omega(t)}(\mathbf{u}^*) - \mathbf{q}(t) + \nabla \cdot \big((1 - \omega(t))\mathbf{C}\varepsilon^{cr}(t)\big)\Big). \quad (57)$$

Estimate (55) follows from (45).

Combining (49), (50), and (51) we note that

$$\|\mathcal{L}_{\omega^1}^{-1}\mathcal{L}_{\omega^1}\mathbf{u}^* - \mathcal{L}_{\omega^2}^{-1}\mathcal{L}_{\omega^2}\mathbf{u}^*\| \leq C\ \|\mathbf{u}^*\|_{V_p}\|\omega^1 - \omega^2\|_{Y_p}. \quad (58)$$

Note also that

$$\|\mathcal{L}_{\omega^1}^{-1}\mathbf{g}^1 - \mathcal{L}_{\omega^2}^{-1}\mathbf{g}^2\| \leq C\|\omega^1 - \omega^1\|_{Y_p}\|\mathbf{g}^1\|_{X_p} + C\|\mathbf{g}^1 - \mathbf{g}^2\|_{X_p},\ \forall \mathbf{g}^1, \mathbf{g}^2 \in X_p. \quad (59)$$

Substituting $\Big(-\mathbf{q}^i(t) + \nabla \cdot \big((1 - \omega^i(t))\mathbf{C}\varepsilon^{cri}(t)\Big)$ for $\mathbf{g}^i$ in (59) and combining (59) with (58), we get (56). This completes the proof of *Lemma 5.3* ■

5.2 Evolution of ω and ε^{cr} with a given u

Let us first analyze evolution operators $\mathcal{R}$, $\mathcal{S}$, which are defined by (25), (26).

Lemma 5.4

There is a constant $C_{5.4}$ with the properties to follow. For all $i, j \in \{1, 2\}$, $k \in \{0, 1, ..., 7\}$, $\rho^1, \rho^2 \in \mathbb{R}^7$ such that $\rho^1_7, \rho^2_7 \leq 1 - \frac{\beta_1}{2}$ we have the following estimates

$$|D_k\mathcal{R}_{i,j}(\rho^1)| + |D_k\mathcal{S}(\rho^1)| \leq C_{5.4}(1 + |\rho^1|)^{\max(m,n)}, \tag{60}$$

$$\begin{aligned}|D_k\mathcal{R}_{i,j}(\rho^1) - D_k\mathcal{R}_{i,j}(\rho^2)| + |D_k\mathcal{S}(\rho^1) - D_k\mathcal{S}(\rho^2)|& \\ \leq C_{5.4}(1 + |\rho^1| + |\rho^2|)^{\max(m,n)}|\rho^1 - \rho^2|.&\end{aligned} \tag{61}$$

Here $D_k = \frac{\partial}{\partial\rho_k}$ for $k \in \{1, 2, ..., 7\}$ and D_0 is the identity operator.

Proof. First let us note that for all $q > 2$

$$|P^q(z)| \leq C|z|^q, \quad |P^q(z) - P^q(z')| \leq C(|z| + |z'|)^{q-1}|z - z'|, \tag{62}$$

$$|\nabla_z P^q(z)| \leq C|z|^{q-1}, \quad |\nabla_z P^q(z) - \nabla_z P^q(z')| \leq C(|z| + |z'|)^{q-2}|z - z'|, \tag{63}$$

where $P(z)$ is defined by (31).

We also note that for all $\rho^1, \rho^2 \in \mathbb{R}^7$, $i, j \in \{1, 2\}$, $k \in \{1, 2, ..., 7\}$

$$|\sigma_{i,j}(\rho)| \leq C|\rho|, \quad |\sigma_{i,j}(\rho^1) - \sigma_{i,j}(\rho^2)| \leq C(1 + |\rho^1| + |\rho^2|)|\rho^1 - \rho^2|, \tag{64}$$

$$|\frac{\partial}{\partial\rho_k}\sigma_{i,j}(\rho)| \leq C(1 + |\rho|), \quad |\frac{\partial}{\partial\rho_k}(\sigma_{i,j}(\rho^1) - \sigma_{i,j}(\rho^2))| \leq C|\rho^1 - \rho^2|. \tag{65}$$

Here $\sigma_{i,j}(\rho)$ is defined by (30).

The lemma is proved after some simple computations. Let us prove for example that

$$|D_k\mathcal{S}(\rho^1) - D_k\mathcal{S}(\rho^2)| \leq C\ (1 + |\rho^1| + |\rho^2|)^m\ |\rho^1 - \rho^2|. \tag{66}$$

We remark that $\mathcal{S}(\rho)$ has the form

$$\mathcal{S}(\rho) = P^m(\sigma_{ij}(\rho))F(\rho_7) \tag{67}$$

with $F \in C^\infty[0, 1 - \frac{\beta_1}{2}]$. Here $\sigma_{ij} = (\sigma_{1,1}, \sigma_{2,2}, \sigma_{1,2})^T$.

$$\begin{aligned}&|D_k\mathcal{S}(\rho^1) - D_k\mathcal{S}(\rho^2)| \\ &\qquad \leq |D_k\big(P^m(\sigma_{ij}(\rho^1))F(\rho^1_7) - P^m(\sigma_{ij}(\rho^2))F(\rho^1_7)\big)| \\ &\qquad + |D_k\big(P^m(\sigma_{ij}(\rho^2))F(\rho^1_7) - P^m(\sigma_{ij}(\rho^2))F(\rho^2_7)\big)| \\ &\leq C\ |D_k\big(P^m(\sigma_{ij}(\rho^1)) - P^m(\sigma_{ij}(\rho^2))\big)| + C\ |P^m(\sigma_{ij}(\rho^1)) - P^m(\sigma_{ij}(\rho^2))| \\ &\quad + C\ |D_kP^m(\sigma_{ij}(\rho^2))| \cdot |\rho^1 - \rho^2| + C\ |P^m(\sigma_{ij}(\rho^2))| \cdot |\rho^1 - \rho^2| \\ &= A + B + C + D.\end{aligned} \tag{68}$$

We abbreviate

$$D_k\sigma(\rho) := D_k(\sigma_{1,1}(\rho), \sigma_{2,2}(\rho), \sigma_{1,2}(\rho))^T. \tag{69}$$

Further,

$$\begin{aligned}
A = C\ &|\nabla_z P^m(\sigma_{ij}(\rho^1))D_k\sigma(\rho^1) - \nabla_z P^m(\sigma_{ij}(\rho^2))D_k\sigma(\rho^2)| \\
&\le C\ |\nabla_z\big(P^m(\sigma_{ij}(\rho^1)) - P^m(\sigma_{ij}(\rho^2))\big)| \cdot |D_k\sigma(\rho^1)| \\
&\quad + C\ |\nabla_z P^m(\sigma_{ij}(\rho^2))| \cdot |D_k\big(\sigma(\rho^1) - \sigma(\rho^2)\big)| \\
&\le C\ \big(|\sigma_{ij}(\rho^1)| + |\sigma_{ij}(\rho^2)|\big)^{m-2} \cdot |\sigma_{ij}(\rho^1) - \sigma_{ij}(\rho^2)| \cdot |D_k\sigma(\rho^1)| \\
&\quad + C\ |\sigma_{ij}(\rho^2)|^{m-1} \cdot |D_k\big(\sigma(\rho^1) - \sigma(\rho^2)\big)| \\
&\le C\ (1 + |\rho^1| + |\rho^2|)^m\ |\rho^1 - \rho^2|. \qquad (70)
\end{aligned}$$

In the same way we obtain

$$\max(B, C, D) \le C\ (1 + |\rho^1| + |\rho^2|)^m\ |\rho^1 - \rho^2|. \qquad (71)$$

Combining this with (70) we get (66). The lemma is proved ■

Lemma 5.5

There is a constant $C_{5.5}$ with the following properties. For all $M, T > 0$, $\mathbf{u}^1, \mathbf{u}^2 \in C^0([0,T], V_p)$ with $\|u^l\|_{V_p,\infty} \le M$ for $l \in \{1,2\}$; $\boldsymbol{\varepsilon}^{cr1}, \boldsymbol{\varepsilon}^{cr2} \in C^0([0,T], Y_p^4)$; $\omega^1, \omega^2 \in C^0([0,T], Y_p)$ such that

$$\omega(x,t) \le 1 - \frac{\beta_1}{2} \quad \forall\ (x,t) \in \Omega \times [0,T], l \in \{1,2\} \qquad (72)$$

$$\|\boldsymbol{\varepsilon}^{crl}(t) - \boldsymbol{\varepsilon}^{crl}(0)\|_{Y_p^4} + \|\omega^l(t) - \omega^l(0)\|_{Y_p} \le 1 \quad \forall\ t \in [0,T], l \in \{1,2\}. \qquad (73)$$

We abbreviate (recall (23))

$$\rho^l := \rho(\mathbf{u}^l, \boldsymbol{\varepsilon}^{crl}, \omega^l). \qquad (74)$$

Then

$$\mathcal{R}(\rho^l)(t) \in Y_p^4, \quad \mathcal{S}(\rho^l)(t) \in Y_p, \quad \forall\ t \in [0,T], l \in \{1,2\}, \qquad (75)$$

$$\|\mathcal{R}(\rho^l)\|_{Y_p^4,\infty} + \|\mathcal{S}(\rho^l)\|_{Y_p,\infty} \le C_{5.5}\big(M + \|\boldsymbol{\varepsilon}^{crl}(0)\|_{Y_p^4} + \|\omega^l(0)\|_{Y_p} + 1\big)^{\max(m,n)+1} \qquad (76)$$

for $l \in \{1,2\}$, and

$$\begin{aligned}
&\|\mathcal{R}(\rho^1) - \mathcal{R}(\rho^2)\|_{Y_p^4,\infty} + \|\mathcal{S}(\rho^1) - \mathcal{S}(\rho^2)\|_{Y_p,\infty} \\
&\quad \le C_{5.5}\Big(M + \sum_{l=1}^{2}\big(\|\boldsymbol{\varepsilon}^{crl}(0)\|_{Y_p^4} + \|\omega^l(0)\|_{Y_p}\big) + 1\Big)^{\max(m,n)+1} \\
&\qquad \cdot \Big(\|\mathbf{u}^1 - \mathbf{u}^2\|_{V_p,\infty} + \|\boldsymbol{\varepsilon}^{cr1} - \boldsymbol{\varepsilon}^{cr2}\|_{Y_p^4,\infty} + \|\omega^1 - \omega^2\|_{Y_p,\infty}\Big). \qquad (77)
\end{aligned}$$

Proof. This lemma is proved by *Corollary 3.1* and *Lemma 5.4*. Let us estimate, for example, the value $\|K\|_{L_p(\Omega)}(t)$, where

$$K(x,t) := \Big|\frac{\partial}{\partial x_r}\big(\mathcal{R}_{i,j}(\rho^1) - \mathcal{R}_{i,j}(\rho^2)\big)(x,t)\Big|, \qquad (78)$$

with $r, i, j \in \{1, 2\}$, $(x, t) \in \Omega \times [0, T]$. We obtain

$$\begin{aligned}
K(x,t) \le \sum_{k=1}^{7} \Big| D_k\mathcal{R}_{i,j}\big(\rho_k^1(x,t)\big)\frac{\partial \rho_k^1(x,t)}{\partial x_r} - D_k\mathcal{R}_{i,j}\big(\rho_k^2(x,t)\big)\frac{\partial \rho_k^2(x,t)}{\partial x_r}\Big| \\
\le \sum_{k=1}^{7} \Big| D_k\mathcal{R}_{i,j}\big(\rho^1(x,t)\big) - D_k\mathcal{R}_{i,j}\big(\rho^2(x,t)\big)\Big| \cdot \Big|\frac{\partial \rho_k^1(x,t)}{\partial x_r}\Big| \\
+ \sum_{k=1}^{7} \Big| D_k\mathcal{R}_{i,j}\big(\rho^2(x,t)\big)\Big| \cdot \Big|\frac{\partial \rho_k^1(x,t)}{\partial x_r} - \frac{\partial \rho_k^2(x,t)}{\partial x_r}\Big| \\
\le \sum_{k=1}^{7} A_k \cdot B_k + C_k \cdot D_k.
\end{aligned} \tag{79}$$

But by *Lemma 5.4* and by *Corollary 3.1* we get

$$\begin{aligned}
A_k \le C\ \big(1 + |\rho^1(x,t)| + |\rho^2(x,t)|\big)^{\max(m,n)} \big|\rho^1(x,t) - \rho^2(x,t)\big| \\
\le C\ \big(1 + \|\rho^1\|_{Y_p^7,\infty} + \|\rho^2(x,t)\|_{Y_p^7,\infty}\big)^{\max(m,n)} \|\rho^1 - \rho^2\|_{Y_p^7,\infty}.
\end{aligned} \tag{80}$$

We get by the same argument

$$C_k \le C\ \big|D_k\mathcal{R}_{i,j}\big(\rho^2(x,t)\big)\big| \le C\ \big(1 + \|\rho^2\|_{Y_p^7,\infty}\big)^{\max(m,n)}. \tag{81}$$

Evidently,

$$\|B_k\|_{L_p} = \|\frac{\partial \rho_k^1(x,t)}{\partial x_r}\|_{L_p} \le \|\rho^1\|_{Y_p^7,\infty}, \tag{82}$$

$$\|D_k\|_{L_p} = \|\frac{\partial \rho_k^1(x,t)}{\partial x_r} - \frac{\partial \rho_k^2(x,t)}{\partial x_r}\|_{L_p} \le \|\rho^1 - \rho^2\|_{Y_p^7,\infty}. \tag{83}$$

Hence

$$\begin{aligned}
\|K\|_{L_p(\Omega)}(t) \le \sum_{k=1}^{7} \|A_k \cdot B_k + C_k \cdot D_k\|_{L_p} \\
\le \sum_{k=1}^{7} \|A_k\|_{C^0} \cdot \|B_k\|_{L_p} + \|C_k\|_{C^0} \cdot \|D_k\|_{L_p} \\
\le C\ \big(1 + \|\rho^1\|_{Y_p^7,\infty} + \|\rho^2\|_{Y_p^7,\infty}\big)^{\max(m,n)+1} \cdot \|\rho^1 - \rho^2\|_{Y_p^7,\infty}.
\end{aligned} \tag{84}$$

It remains to check that

$$\|\rho^1 - \rho^2\|_{Y_p^7,\infty} \le C\ \Big(\|\mathbf{u}^1 - \mathbf{u}^2\|_{V_p,\infty} + \|\varepsilon^{cr1} - \varepsilon^{cr2}\|_{Y_p^4,\infty} + \|\omega^1 - \omega^2\|_{Y_p,\infty}\Big), \tag{85}$$

and

$$\big(1 + \|\rho^1\|_{Y_p^7,\infty} + \|\rho^2\|_{Y_p^7,\infty}\big) \le C\ \Big(M + \sum_{l=1}^{2} \big(\|\varepsilon^{crl}(0)\|_{Y_p^4} + \|\omega^l(0)\|_{Y_p}\big) + 1\Big). \tag{86}$$

This concludes the proof of *Lemma 5.5* ■

We now prove the existence of the solution $(\boldsymbol{\varepsilon}^{cr}, \omega) \in C^0([0,T], Y_p^4 \times Y_p)$ if the displacement field $\mathbf{u} \in C^0([0,T], V_p)$ is given.

Lemma 5.6

Let $M > 0$, $\boldsymbol{\varepsilon}_0^{cr} \in Y_p^4$, and $\omega_0 \in Y_p^{\beta_1,\beta_2}$. Put

$$T_0 := \left[C_{5.5}\big(M + 2\|\boldsymbol{\varepsilon}_0^{cr}\|_{Y_p^4} + 2\|\omega_0\|_{Y_p} + 1\big)^{\max(m,n)+1} 2\frac{1 + C_I}{\min(\beta_1, 2\beta_2)}\right]^{-1} \tag{87}$$

with $C_{5.5}$ from *Lemma 5.5* and C_I from *Corollary 3.1.* Let $T' \in (0, T_0]$ and $\mathbf{u} \in C^0([0,T'], V_p)$, such that $\|\mathbf{u}\|_{V_p,\infty} \le M$. Then there exists a unique mapping $(\boldsymbol{\varepsilon}^{cr}, \omega) = \big(\boldsymbol{\varepsilon}^{cr}, \omega\big)(\mathbf{u}, \boldsymbol{\varepsilon}_0^{cr}, \omega_0) \in C^0([0,T'], Y_p^4 \times Y_p)$ such that

$$(\boldsymbol{\varepsilon}^{cr}, \omega)(t) = (\boldsymbol{\varepsilon}_0^{cr}, \omega_0) + \int_0^t \big(\mathcal{R}(\rho(\mathbf{u}, \boldsymbol{\varepsilon}^{cr}, \omega)), \mathcal{S}(\rho(\mathbf{u}, \boldsymbol{\varepsilon}^{cr}, \omega))\big)(s)ds, \tag{88}$$

$$\|\boldsymbol{\varepsilon}^{cr}(t) - \boldsymbol{\varepsilon}^{cr}(0)\|_{Y_p^4} + \|\omega(t) - \omega(0)\|_{Y_p} \le \frac{\min(\beta_1, 2\beta_2)}{2(1 + C_I)} \quad \forall\, t \in [0, T']. \tag{89}$$

Moreover

$$\omega(t) \in Y_p^{\frac{\beta_1}{2}, 2\beta_2} \quad \forall\, t \in [0, T'], \tag{90}$$

$$(\boldsymbol{\varepsilon}^{cr}, \omega) \in C^1([0,T'], Y_p^4 \times Y_p), \tag{91}$$

$$(\dot{\boldsymbol{\varepsilon}^{cr}}, \dot{\omega})(t) = \big(\mathcal{R}(\rho(\mathbf{u}, \boldsymbol{\varepsilon}^{cr}, \omega)), \mathcal{S}(\rho(\mathbf{u}, \boldsymbol{\varepsilon}^{cr}, \omega))\big)(t) \quad \forall\, t \in (0, T'). \tag{92}$$

Proof. As it is done in [2] for partly coupled damage model, we adapt the standard proof of the existence of solutions to ordinary differential equations in Banach spaces. We define the closed subset of $C^0([0,T'], Y_p^4 \times Y_p)$ by

$$\begin{aligned}\mathcal{M} := \Big\{ &(\boldsymbol{\varepsilon}^{cr}, \omega) \in C^0([0,T'], Y_p^4 \times Y_p) : (\boldsymbol{\varepsilon}^{cr}, \omega)(0) = (\boldsymbol{\varepsilon}_0^{cr}, \omega_0), \\ &\|\boldsymbol{\varepsilon}^{cr}(t) - \boldsymbol{\varepsilon}^{cr}(0)\|_{Y_p^4} + \|\omega(t) - \omega(0)\|_{Y_p} \le \frac{\min(\beta_1, 2\beta_2)}{2(1 + C_I)} \Big\}.\end{aligned} \tag{93}$$

The application of *Corollary 3.1* yields for every $(\boldsymbol{\varepsilon}^{cr}, \omega) \in \mathcal{M}$

$$\omega(t) \in Y_p^{\frac{\beta_1}{2}, 2\beta_2} \quad \forall\, t \in [0, T']. \tag{94}$$

Note that for all $t \in [0, T'], (\boldsymbol{\varepsilon}^{cr}, \omega) \in \mathcal{M}$

$$\|\boldsymbol{\varepsilon}^{cr}(t) - \boldsymbol{\varepsilon}^{cr}(0)\|_{Y_p^4} + \|\omega(t) - \omega(0)\|_{Y_p} \le \frac{\min(\beta_1, 2\beta_2)}{2(1 + C_I)} \le 1. \tag{95}$$

Hence, by *Lemma 5.5*, we obtain for $t \leq t', (\boldsymbol{\varepsilon}^{cr}, \omega) \in \mathcal{M}$

$$\int_{t}^{t'} \left\| \big(\mathcal{R}(\rho(\mathbf{u}, \boldsymbol{\varepsilon}^{cr}, \omega)), \mathcal{S}(\rho(\mathbf{u}, \boldsymbol{\varepsilon}^{cr}, \omega))\big)(s) \right\|_{Y_p^4 \times Y_p} ds$$
$$\leq C_{5.5} \big(M + \|\boldsymbol{\varepsilon}^{crl}(0)\|_{Y_p^4} + \|\omega^l(0)\|_{Y_p} + 1 \big)^{\max(m,n)+1} (t - t')$$
$$\leq \frac{\min(\beta_1, 2\beta_2)}{2(1 + C_I)}. \quad (96)$$

Let the mapping $\mathcal{T} : \mathcal{M} \to C^0([0, T'], Y_p^4 \times Y_p)$ be given by

$$\mathcal{T}(\boldsymbol{\varepsilon}^{cr}, \omega)(t)$$
$$= (\boldsymbol{\varepsilon}_0^{cr}, \omega_0) + \int_{0}^{t} \big(\mathcal{R}(\rho(\mathbf{u}, \boldsymbol{\varepsilon}^{cr}, \omega)), \mathcal{S}(\rho(\mathbf{u}, \boldsymbol{\varepsilon}^{cr}, \omega))\big)(s) ds \quad \forall\, t \in [0, T']. \quad (97)$$

Accordingly to (96), this mapping is well defined. Taking into account (96), we obtain

$$\|\mathcal{T}(\boldsymbol{\varepsilon}^{cr}, \omega)(t) - (\boldsymbol{\varepsilon}_0^{cr}, \omega_0)\|_{Y_p^4 \times Y_p} \leq \frac{\min(\beta_1, 2\beta_2)}{2(1 + C_I)} \quad \forall\, t \in [0, T']. \quad (98)$$

Therefore, $\mathcal{T}(\mathcal{M}) \subset \mathcal{M}$. By *Lemma 5.5*, it follows that $\mathcal{T}$ is a contraction with respect to the norm of the space $C^0([0, T'], Y_p^4 \times Y_p)$. Indeed, for every instant of time t we have

$$\|\mathcal{T}(\boldsymbol{\varepsilon}^{cr1}, \omega^1)(t) - \mathcal{T}(\boldsymbol{\varepsilon}^{cr2}, \omega^2)(t)\|_{Y_p^4 \times Y_p}$$
$$\leq C_{5.5} \Big(M + \sum_{l=1}^{2} \big(\|\boldsymbol{\varepsilon}^{crl}(0)\|_{Y_p^4} + \|\omega^l(0)\|_{Y_p} \big) + 1 \Big)^{\max(m,n)+1}$$
$$\cdot \Big(\|\boldsymbol{\varepsilon}^{cr1} - \boldsymbol{\varepsilon}^{cr2}\|_{Y_p^4, \infty} + \|\omega^1 - \omega^2\|_{Y_p, \infty} \Big) T' \leq$$
$$\frac{1}{4} \Big(\|\boldsymbol{\varepsilon}^{cr1} - \boldsymbol{\varepsilon}^{cr2}\|_{Y_p^4, \infty} + \|\omega^1 - \omega^2\|_{Y_p, \infty} \Big). \quad (99)$$

It follows from *Banach's fixed point theorem* (see, for example, [6]) that there exists a unique $(\boldsymbol{\varepsilon}^{cr}, \omega) \in \mathcal{M}$ such that $\mathcal{T}(\boldsymbol{\varepsilon}^{cr}, \omega) = (\boldsymbol{\varepsilon}^{cr}, \omega)$. Thus, we have proved that the pair $(\boldsymbol{\varepsilon}^{cr}, \omega) \in C^0([0, T'], Y_p^4 \times Y_p)$ is uniquely defined by (88), (89).

To conclude the proof it remains to note that the mapping $t \mapsto \big(\mathcal{R}(\rho(\mathbf{u}, \boldsymbol{\varepsilon}^{cr}, \omega)), \mathcal{S}(\rho(\mathbf{u}, \boldsymbol{\varepsilon}^{cr}, \omega))\big)(t)$ is continuous from $[0, T']$ to $Y_p^4 \times Y_p$. Since (90) holds, we may use *Lemma 5.4* to prove that

$$\left\| \big(\mathcal{R}(\rho(\mathbf{u}, \boldsymbol{\varepsilon}^{cr}, \omega)), \mathcal{S}(\rho(\mathbf{u}, \boldsymbol{\varepsilon}^{cr}, \omega))\big)(t^1) \right.$$
$$\left. - \big(\mathcal{R}(\rho(\mathbf{u}, \boldsymbol{\varepsilon}^{cr}, \omega)), \mathcal{S}(\rho(\mathbf{u}, \boldsymbol{\varepsilon}^{cr}, \omega))\big)(t^2) \right\|_{Y_p^4 \times Y_p} \to 0, \quad \text{as } t^1 \to t^2. \quad (100)$$

Lemma 5.6 is proved ■

Now we need to estimate the difference between two solutions of (88), (89). Let us abbreviate

$$\hat{C} = C_{5.5}\big(M + 2\|\boldsymbol{\varepsilon}_0^{cr}\|_{Y_p^4} + 2\|\omega_0\|_{Y_p} + 1\big)^{\max(m,n)+1} \tag{101}$$

with $C_{5.5}$ from *Lemma 5.5* and C_I from *Corollary 3.1*.

Lemma 5.7

Let $M > 0$, $K \geq 1$, $\boldsymbol{\varepsilon}_0^{cr} \in Y_p^4$, and $\omega_0 \in Y_p^{\beta_1,\beta_2}$. Put

$$T_1 := \Big[\hat{C} \cdot 2\max(K, \frac{1 + C_I}{\min(\beta_1, 2\beta_2)})\Big]^{-1} \tag{102}$$

where $\hat{C}$ is given by (101). Let $T' \in (0, T_1]$, and $\mathbf{u}^1, \mathbf{u}^2 \in C^0([0, T'], V_p)$, such that $\|\mathbf{u}^l\|_{V_p,\infty} \leq M$ for $l \in \{1, 2\}$. Assume that

$$(\boldsymbol{\varepsilon}^{cr l}, \omega^l) = \big(\boldsymbol{\varepsilon}^{cr}, \omega\big)(\mathbf{u}^l, \boldsymbol{\varepsilon}_0^{cr}, \omega_0) \in C^0([0, T'], Y_p^4 \times Y_p), l \in \{1, 2\} \tag{103}$$

are defined by (88), (89). Then

$$\|\boldsymbol{\varepsilon}^{cr1} - \boldsymbol{\varepsilon}^{cr2}\|_{Y_p^4,\infty} + \|\omega^1 - \omega^2\|_{Y_p,\infty} \leq \frac{1}{K}\|\mathbf{u}^1 - \mathbf{u}^2\|_{V_p,\infty}. \tag{104}$$

Proof. By *Lemma 5.5* we have

$$\begin{aligned}
&\|\mathcal{R}(\rho^1) - \mathcal{R}(\rho^2)\|_{Y_p^4,\infty} + \|\mathcal{S}(\rho^1) - \mathcal{S}(\rho^2)\|_{Y_p,\infty} \\
&\qquad \leq \hat{C} \cdot \Big(\|\mathbf{u}^1 - \mathbf{u}^2\|_{V_p,\infty} + \|\boldsymbol{\varepsilon}^{cr1} - \boldsymbol{\varepsilon}^{cr2}\|_{Y_p^4,\infty} + \|\omega^1 - \omega^2\|_{Y_p,\infty}\Big).
\end{aligned} \tag{105}$$

Therefore, since (88) holds, we obtain

$$\begin{aligned}
&\|\boldsymbol{\varepsilon}^{cr1} - \boldsymbol{\varepsilon}^{cr2}\|_{Y_p^4,\infty} + \|\omega^1 - \omega^2\|_{Y_p,\infty} \\
&\qquad \leq \hat{C}T'\Big(\|\mathbf{u}^1 - \mathbf{u}^2\|_{V_p,\infty} + \|\boldsymbol{\varepsilon}^{cr1} - \boldsymbol{\varepsilon}^{cr2}\|_{Y_p^4,\infty} + \|\omega^1 - \omega^2\|_{Y_p,\infty}\Big) \\
&\qquad \leq \frac{1}{2K}\|\mathbf{u}^1 - \mathbf{u}^2\|_{V_p,\infty} + \frac{1}{2}\Big(\|\boldsymbol{\varepsilon}^{cr1} - \boldsymbol{\varepsilon}^{cr2}\|_{Y_p^4,\infty} + \|\omega^1 - \omega^2\|_{Y_p,\infty}\Big).
\end{aligned} \tag{106}$$

Now inequality (104) follows from (106). *Lemma 5.7* is proved ■

5.3 Equilibrium equations coupled with evolution law

In this subsection we solve problem (32), (33) and prove *Theorem 4.1*.

Proof of Theorem 4.1. We choose T_1 as in *Lemma 5.7* where we put

$$K := 2 \cdot C_{5.3}\big(\|\mathbf{u}^*\|_{V_p} + \|\boldsymbol{\varepsilon}_0^{cr}\|_{V_p^4} + \|\mathbf{q}\|_{X_p,\infty} + 1\big), \tag{107}$$

$$M := C_{5.3}\big(\|\mathbf{u}^*\|_{V_p} + \|\boldsymbol{\varepsilon}_0^{cr}\|_{V_p^4} + \|\mathbf{q}\|_{X_p,\infty} + 1\big) + \|\mathbf{u}^*\|_{V_p} \tag{108}$$

with $C_{5.3}$ from *Lemma 5.3*. In order to use *Banach's fixed point theorem* we define the closed subset of $C^0([0, T'], V_p^0)$ by

$$\mathcal{M} := \Big\{\mathbf{u} \in C^0([0, T'], V_p^0) : \|\mathbf{u} + \mathbf{u}^*\|_{V_p,\infty} \leq M\Big\}. \tag{109}$$

Let the mapping $\mathcal{T} : \mathcal{M} \to C^0([0,T'], V_p^0)$ be given by

$$\mathcal{T}(\mathbf{u}) = \mathbf{U}\big(\mathbf{u}^*, \boldsymbol{\varepsilon}^{cr}(\mathbf{u}+\mathbf{u}^*, \boldsymbol{\varepsilon}_0^{cr}, \omega_0), \omega(\mathbf{u}+\mathbf{u}^*, \boldsymbol{\varepsilon}_0^{cr}, \omega_0), \mathbf{q}\big), \tag{110}$$

where $\big(\boldsymbol{\varepsilon}^{cr}, \omega\big)(\mathbf{u}, \boldsymbol{\varepsilon}_0^{cr}, \omega_0)$ is defined by (88) and $\mathbf{U}\big(\mathbf{u}^*, \boldsymbol{\varepsilon}^{cr}, \omega, \mathbf{q}\big)$ is introduced in *Lemma 5.3*.

Let us show that $\mathcal{T}(\mathcal{M}) \subset C^0([0,T'], V_p^0)$. In fact, accordingly to *Lemma 5.6*, for all $\mathbf{u} \in \mathcal{M}$ we have $\big(\boldsymbol{\varepsilon}^{cr}, \omega\big)(\mathbf{u}+\mathbf{u}^*, \boldsymbol{\varepsilon}_0^{cr}, \omega_0) \in C^0([0,T'], Y_p^4 \times Y_p)$. Therefore, the assertion follows from *Lemma 5.3* and the mapping $\mathcal{T}$ is well defined.

Now let us show that $\mathcal{T}(\mathcal{M}) \subset \mathcal{M}$. Using (55) and (89) we obtain for $\mathbf{u} \in \mathcal{M}$

$$\begin{aligned} &\|\mathcal{T}(\mathbf{u}) + \mathbf{u}^*\|_{V_p,\infty} \le \|\mathcal{T}(\mathbf{u})\|_{V_p,\infty} + \|\mathbf{u}^*\|_{V_p} \\ &\quad \le C_{5.3}\big(\|\mathbf{u}^*\|_{V_p} + \|\boldsymbol{\varepsilon}^{cr}(\mathbf{u}+\mathbf{u}^*, \boldsymbol{\varepsilon}_0^{cr}, \omega_0)\|_{Y_p^4,\infty} + \|\mathbf{q}\|_{X_p,\infty}\big) + \|\mathbf{u}^*\|_{V_p} \le M. \end{aligned} \tag{111}$$

Let us prove that $\mathcal{T}$ is a contraction. Taking into account (56), (104), and the choice of K, we obtain

$$\begin{aligned} &\|\mathcal{T}(\mathbf{u}^1) - \mathcal{T}(\mathbf{u}^2)\|_{V_p,\infty} \le \\ &\qquad C_{5.3}\Big(\|\omega^1 - \omega^2\|_{Y_p,\infty}\big(\|\mathbf{u}^*\|_{V_p} + \|\boldsymbol{\varepsilon}_0^{cr}\|_{Y_p^4} + \|\mathbf{q}\|_{X_p,\infty} + 1\big) \\ &\qquad\qquad + \|\boldsymbol{\varepsilon}^{cr1} - \boldsymbol{\varepsilon}^{cr2}\|_{Y_p^4,\infty}\Big) \\ &\qquad \le \frac{C_{5.3}}{K}\big(\|\mathbf{u}^*\|_{V_p} + \|\boldsymbol{\varepsilon}_0^{cr}\|_{Y_p^4} + \|\mathbf{q}\|_{X_p,\infty} + 1\big)\|\mathbf{u}^1 - \mathbf{u}^2\|_{V_p,\infty} \\ &\qquad\qquad \le \frac{1}{2}\|\mathbf{u}^1 - \mathbf{u}^2\|_{V_p,\infty}. \end{aligned} \tag{112}$$

From *Banach's fixed point theorem* it follows that there is a uniquely determined mapping $\mathbf{u} \in \mathcal{M}$, such that $\mathcal{T}(\mathbf{u}) = \mathbf{u}$. To obtain the mapping $(\mathbf{u}, \boldsymbol{\varepsilon}^{cr}, \omega) \in C^0([0,T'], V_p^0 \times Y_p^4 \times Y_p)$ that satisfy (32) and (33), we put (see *Lemma 5.6*)

$$\boldsymbol{\varepsilon}^{cr} := \boldsymbol{\varepsilon}^{cr}(\mathbf{u}+\mathbf{u}^*, \boldsymbol{\varepsilon}_0^{cr}, \omega_0), \quad \omega := \omega(\mathbf{u}+\mathbf{u}^*, \boldsymbol{\varepsilon}_0^{cr}, \omega_0). \tag{113}$$

The existence of the solution to (32), (33) is proved. Although the uniqueness in $\mathcal{M}$ is guarantied by *Banach's fixed point theorem*, it remains to check that the solution is uniquely determined by (32), (33). Assume the converse, then there are two different solutions of (32), (33)

$$(\mathbf{u}^l, \boldsymbol{\varepsilon}^{crl}, \omega^l) \in C^0([0,T'], V_p^0 \times Y_p^4 \times Y_p), \quad l \in \{1,2\}. \tag{114}$$

Put

$$t_0 := \max\{\hat{t} \in [0,T'] : (\mathbf{u}^1, \boldsymbol{\varepsilon}^{cr1}, \omega^1)(t) = (\mathbf{u}^2, \boldsymbol{\varepsilon}^{cr2}, \omega^2)(t) \quad \text{for } t \in [0,\hat{t}]\}. \tag{115}$$

Hence,

$$(\mathbf{u}^1, \boldsymbol{\varepsilon}^{cr1}, \omega^1)(t) = (\mathbf{u}^2, \boldsymbol{\varepsilon}^{cr2}, \omega^2)(t) \quad \text{for } t \in [0,t_0]. \tag{116}$$

Moreover, $t_0 < T'$ and for every $\check{t} \in (t_0, T']$ there exists $\tilde{t} \in (t_0, \check{t}]$, such that

$$(\mathbf{u}^1, \boldsymbol{\varepsilon}^{cr1}, \omega^1)(\tilde{t}) \neq (\mathbf{u}^2, \boldsymbol{\varepsilon}^{cr2}, \omega^2)(\tilde{t}). \tag{117}$$

Arguing as above, we prove the uniqueness of the solution $(\mathbf{u}, \boldsymbol{\varepsilon}^{cr}, \omega) \in C^0([t_0, T^{new\prime}], V_p^0 \times Y_p^4 \times Y_p)$ to the new problem

$$\boldsymbol{\nabla} \cdot \big((1-\omega)\mathbf{C}(\boldsymbol{\varepsilon}(\mathbf{u}+\mathbf{u}^*) - \boldsymbol{\varepsilon}^{cr})\big) = -\mathbf{q}(t) \quad \forall\, t \in [t_0, T^{new\prime}], \tag{118}$$

$$(\boldsymbol{\varepsilon}^{cr}, \omega)(t) = (\boldsymbol{\varepsilon}^{cr1}(t_0), \omega^1(t_0)) + \int_{t_0}^{t} \big(\mathcal{R}(\rho(\mathbf{u}+\mathbf{u}^*, \boldsymbol{\varepsilon}^{cr}, \omega), \mathcal{S}(\rho(\mathbf{u}+\mathbf{u}^*, \boldsymbol{\varepsilon}^{cr}, \omega)\big)(s)ds \quad \forall\, t \in [t_0, T^{new\prime}] \tag{119}$$

$$\|\mathbf{u}(t) + \mathbf{u}^*\|_{V_p} \leq M^{new} \quad \forall\, t \in [t_0, T^{new\prime}], \tag{120}$$

with some new parameters $T^{new\prime}, M^{new}$. The reader will easily prove that there exists $\check{t} \in (t_0, \min(T^{new\prime}, T')]$, such that

$$\|\mathbf{u}^l(t) + \mathbf{u}^*\|_{V_p} \leq M^{new} \quad \forall\, t \in [t_0, \check{t}],\ l \in \{1, 2\}. \tag{121}$$

That means, that

$$(\mathbf{u}^1, \boldsymbol{\varepsilon}^{cr1}, \omega^1)(t) = (\mathbf{u}^2, \boldsymbol{\varepsilon}^{cr2}, \omega^2)(t) \quad \forall\, t \in [t_0, \check{t}]. \tag{122}$$

This contradiction proves the theorem ■

6 Conclusions

The creep damage problem is formulated in a well-posed manner. *Theorem 4.1* states that a unique smooth solution to the Kachanov-Rabotnov problem exists in a certain time interval $[0, T_1]$. The corresponding function spaces X_p, Y_p, and V_p reflect the essence of the system of equations and can be used for a proper mathematical analysis of the problem. Particulary, clear definitions of terms "stable", "unstable", and "convergency" can be given.

It is shown that the requirements of the existence theorem (*Theorem 4.1*) have a physical meaning and the violation of these requirements directly affects the lifetime estimate.

If we do not impose any restrictions on the gradient of initial damage (such situation corresponds to $\beta_2 = \infty$), then the lifetime t^* of the structure can be arbitrary small even if $\min(1-\omega_0) \geq \beta_1 > 0$.

This damage localization effect is characterized at each instant of time by the quantity

$$\Lambda(t) = \frac{\|\nabla\omega\|_{L_p}}{\min(1-\omega)}(t). \tag{123}$$

The value $\Lambda(t)$ controls the remaining life of the structure $t_{rest} := t^* - t$.

$$t_{rest} \to 0, \quad \text{as} \quad \Lambda \to \infty. \tag{124}$$

Thus, the estimation of Λ gives an answer to the question when the damage becomes critical. This measure of damage localization can be adopted to improve monitoring and inspection strategies used to secure the reliable operation of engineering structures.

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